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Finding the mean/covariance of a system

When we look at a non-deterministic system, it is important to check the variance and covariance of the output y. It determines how certain we are of our results. When we look at the output, there are two types of question we might encounter. The type is which the transient state (when the system just starts) matters. The second type we only look at the steady state. This assumes that we can let the system warm up for some time t. In this paper we are trying to find the mean/covariance of a system at the steady state.

Let us consider this system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x + v$$

where ω and v are unit strength white noises, x and v are independent. $\zeta = 1/10$ $\omega_n = 2$.

Let's first find the mean of x and y

we know that x can be solved as

$$x(t) = \int_{-\infty}^{t} e^{A(t-\tau)} B\omega(\tau)\, d\tau \quad \forall t \in \mathbb{R}$$

the expectation of $x(t)$ would then be

$$E[x(t)] = E\left[\int_{-\infty}^{t} e^{A(t-\tau)} B\omega(\tau)\, d\tau \quad \forall t \in \mathbb{R}\right]$$

$$\int_{-\infty}^{t} e^{A(t-\tau)} B \quad E[\omega(\tau)] \quad d\tau \quad \forall t \in \mathbb{R}$$

we know that since $E[\omega(\tau)] = 0$

E[x(t)] = 0

now if we want to find the expectation of y

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x + v$$

$$E[y] = E[\begin{pmatrix} 1 & 1 \end{pmatrix} x + v]$$

$$E[y] = \begin{pmatrix} 1 & 1 \end{pmatrix} E[x] + E[v]$$

this is also pretty simple since we already calculated that the expectation of $x=0$ and v is white noise so $E[v]=0$

$$E[y]=0$$

Next we want to find the covariance of x and y

$$\text{Cov}[\ x\ ] = \text{E}[\ x(t)x(t)^*\ ] = \text{ P}$$

P is the solution to the lyapunov equation such that

$$\text{AP} + \text{PA}^* + \text{BWB}^* = 0$$

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad W = 1$$

$$\begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} P + \text{P}\begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix}^* + \begin{pmatrix} 1 \\ 1 \end{pmatrix} 1 \begin{pmatrix} 1 & 1 \end{pmatrix} = 0$$

You can solve for P by hand, but I do not recommend it instead try these code in matlab:

```
z = 1/10.0;
w = 2;
A = [0 1;-w2 -2*z*w];
Q = [1 1;1 1];
lya = lyap(A,Q)
```

And we get the solution P as

```
lya =
    1.8625   -0.5000
   -0.5000    6.2500
```

Next we want to find the Covariance of y

$$\text{Cov}[\,y\,] = \text{E}[\ y(t)y(t)^*] \text{ - E}[\ y(t)\ ]\ \text{E}[\ y(t)\ ]\ ^*$$

since the expectation of y is zero we only have to worry about the first term.

$\mathrm{Cov}[\, y\,] = \mathrm{E}[\; y(t)y(t)^*]$

$\mathrm{E}[\; y(t)y^*(t)] = E\Big[\, ((\begin{smallmatrix} 1 & 1 \end{smallmatrix})x + v)(x^* \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + v^*)\;\Big]$

$\mathrm{E}[\; y(t)y^*(t)] = (\begin{smallmatrix} 1 & 1 \end{smallmatrix})E[x\,x^*]\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + E[\,v^*x^*\,]\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + (\begin{smallmatrix} 1 & 1 \end{smallmatrix})E[x\,v^*] + E[v\,v^*]$

Previously we stated that v and x are independent so their covariance would be zero

$\mathrm{E}[\; y(t)y^*(t)] = (\begin{smallmatrix} 1 & 1 \end{smallmatrix})E[x\,x^*]\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + 1$

$\mathrm{E}[\; y(t)y^*(t)] = (\begin{smallmatrix} 1 & 1 \end{smallmatrix})\left(\begin{smallmatrix} 1.8625 & -0.5 \\ -0.5 & 6.25 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) + 1$

if you add these up in matlab you would get

Cy =

8.1125

Now let's try to find the expectation on this

$$\mathrm{E}\,\left[\lim_{T\to\infty} \frac{1}{T}\int_0^T\; x(t)'\,Q\,x(t)\,dt\right]$$

where Q

$$\mathrm{Q} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$$

using this property

$$\mathrm{E}\,\left[\lim_{T\to\infty} \frac{1}{T}\int_0^T\; z(t)'\,z(t)\,dt\right] = \mathrm{trace}(\mathrm{P}_z)$$

z in this case would be $\mathrm{Q}^{1/2}x(t)$

$z = Q^{1/2}\,x(t) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)^{1/2} x(t)$

$\mathrm{P}_z = E[Q^{1/2}x(t)x^*(t)Q^{*\,1/2}]$

$\mathrm{P}_z = Q^{1/2}\,E[x(t)x^*(t)]Q^{*\,1/2}$

if we put this into matlab we would get

$\mathrm{trace}(\mathrm{P}_z) = 15.225$